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MOTION OF SPHERES IN STILL FLUIDS.

By P. Hirsch.

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MOTION OF SPHERES IN STILL FLUIDS.*

By P. Hirsch.

The behavior of a liquid or gaseous medium, in which a solid body can move freely under the action of a force of constant magnitude and direction, is yet little known. Only in connection with a few special problems, which belong in this field, have experiments been tried and these chiefly concern technically important cases. The behavior of the simplest shaped bodies has as yet been scarcely investigated at all. We began the systematic investigation of this problem with a sphere, not merely because this is the simplest geometrical body, but also because the potential motion around it is known. The chief question, to which an answer is sought in this manner, reads: "At the beginning of the motion, does the so-called

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This paper was accepted by the philosophical faculty of the Göttingen University (Dr. Prandtl, Referent). The experiments were executed during 1909-1911, in the Institute for Applied Mechanics of the Göttingen University. The writer takes pleasure in expressing his heartiest thanks to Dr. Prandtl for the assignment of the task, his constant interest in the progress of the work and for the use of the facilities of the Institute.

'apparent mass'* have the value computed from the potential motion?"

This limitation of the question was due to the circumstance that the experimental apparatus employed enabled the accurate determination of the acceleration even at a vanishingly small velocity, whereas repeated photographing of the sphere on one and the same stationary plate seemed preferable for other parts of the process (Compare the following Leipzig experiments). More accurate information on the details of the flow are to be expected from photographing the surface of water through which a half-submerged sphere is being towed.

One of the most important helps was a kinetograph. The kinetographic measuring devices were made especially for these experiments, a subordinate purpose of which was to determine the availability of kinetographs for acceleration measurements.

<u>1. Leipzig Experiments</u>.- Of experiments already published, only those **performed** by Schiller and Döge need to be men-

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*The "apparent mass" of a body moving in a fluid is the quotient of the external force acting upon it (without taking into account the forces of the fluid itself) divided by the actual acceleration of the center of gravity, less the actual mass of the body. A simple energy consideration shows that the apparent mass is always positive in potential motion. The apparent mass divided by the mass of fluid displaced by the body is a non-dimensional number, the "relative apparent mass" of the body. For spheres in non-viscous fluids it is 0.5. The use of the expression "apparent mass" will be avoided in this paper, when the direction of acceleration of the body differs from that of the external force.

tioned.* They employed pilot balloons, which they photographed, at uniform intervals of about one-half second, on one and the same plate and simultaneously on a second plate located at an angle of about 90° to the other plate. The lifting power and diameter of the balloon were found before and after the experiment and interpolated according to the time. From the velocity attained after a long time they first defined two quantities a and α in the equation: Resistance or drag = $a\ F\ \rho\ v^{\alpha}$, in which F denotes the area of the equator. They then defined an "equivalent mass" as

$$M = \frac{P - a\ F\ \rho\ v^{\alpha}}{\frac{dv}{dt}}$$

The "equivalent mass" computed from the Leipzig experiments

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* Schiller and Döge, "Widerstands- und Beschleunigungsmessungen an frei steigenden Pilotballonen mittels photographischer Registrierung," Physikalische Zeitschrift I, 1912, p.334.

After the present paper had been written, there appeared another paper by F. S. Schmidt, "Zur beschleunigten Bewegung kugelförmiger Körper in widerstehenden Mitteln." Dissertation, Leipzig, 1919. Schmidt's experiments are a continuation of the ones by Schiller and Döge. In addition to pilot balloons, he photographed metal-weighted wax balls falling in water. Among the interesting results, the most important is the observation, in all starts, of a maximum velocity of 0.86 followed by a minimum velocity of 0.815 of the final velocity. By coloring the marginal layer, the dependence of these phenomena on the formation, growth and disappearance of a vortex ring could be determined. Schmidt eliminated, as faulty, all experiments in which there was a strong lateral deviation. Such a selection may result in somewhat different final results in comparison with the present article, in which the lateral deviation is regarded as a peculiarity of the starting process. Hence allowance must be made for the difference in the evaluation of the individual experiments in a later comparison of the two papers.

differs in two respects from the "apparent mass"* defined in the beginning of the present treatise. The actual mass and contents of the sphere are included in the "equivalent mass" but not in the "apparent mass." Moreover, in calculating the equivalent mass, a value is first obtained which corresponds to the resistance or drag for the same but long constant velocity. A potential formula for the magnitude of this resistance was obtained from the vertical velocities attained after a long time in the experiments. There can be no objection to this method, so long as the only object is to express the experimental results in numerical form. It appears questionable, however, to make said deduction from an approximation formula thus obtained, since this means an extrapolation and, indeed, for a function, the complexity of which has been demonstrated experimentally. Aside from this, the objection may be made to the deduction, that it is taken from the conception of an entirely different flow and is therefore calculated to misrepresent the experimental results.

Aside from the immediate friction, the resistance is the integral of the surface pressure produced by the motion of the fluid. Its separation into two components, one of which may be considered as caused by the velocity, and the other by the acceleration, would therefore presuppose a physical presentation enabling the state of motion to be represented as proceeding from

*The expression "apparent mass" was avoided in the Leipzig paper, because it already had a different meaning in physics. On the other hand, attention is called to the fact that in that case it has to do with the apparent mass of an electron in the electric field and in this case with the apparent mass of a body in the hydrodynamic field, hence in both cases with a scalar peculiar to one vector field with a perfectly analogous physical meaning in both cases. The same expression is used, in order to emphasize this analogy.

a uniform motion and a modification of the same due to the acceleration. So long as such a theory is lacking, a separation of the resistance caused by the accelerated motion can only ensue from arbitrary formulas, from which no new insight into the process can be expected (See also Section 7, "The Start").

Only the small component, which represents the integral of the tangential components of the surface forces, can be approximately computed, and that by means of the marginal-layer theory.* A rough estimate, on the basis of the Göttingen experiments, indicates that the frictional resistance in the very first stage of the motion, for a distance of about half the radius of the balloon, is negligible in comparison with the experimental errors (See Section 6).

As regards the accuracy of the spherical shape, the Leipzig experiments are probably superior to the Göttingen experiments and, with the more accurate evaluability of a picture on a stationary photographic plate in comparison with a kinetogram, more information can be obtained on the middle region of the starting process**

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* Prandtl: "Ueber Flüssigkeitsbewegung bei sehr kleinen Reibung." Verhandlungen des III Internationalen Mathematiker-Kongresses, 1904. Leipzig, 1905.

** In the introduction to the paper of Schiller and Döge, reference is made to the practical importance of the investigation of the "equivalent mass" for free-balloon flight. Now, it is precisely the start without lateral wind (both in the Leipzig and in the Göttingen experiments) which is without interest for the free-balloon traveller, who, under such a condition, can start with as small a lifting force as he pleases.

For starting with a wind, all practical requirements will be met by the following consideration, which was given by the author in a lecture before the Lower-Saxony Aviation Club. The tangent to the path of the starting balloon must coincide with the direction of the resultant of all the forces acting on the balloon. These forces are the lifting power of the balloon and the action of the wind on the balloon. The ratio of these two forces is equal to the tangent of the angle at which the balloon leaves the ground.

2. Göttingen Experiments.- The below-described starting experiments with toy balloons were preceded by similar experiments with a free balloon of 1437 m^3 (50747 cu.ft.). These experiments were tried in connection with two ascents made under the management of the Lower-Saxony Aviation Club. The vertical distance flown was measured in one instance by means of a kinetograph on the ground and in the other instance by means of a barograph on board. The failure of these experiments to give any useful results was due first to the uncertain determination of the lifting power, and secondly to the disturbed condition of the atmosphere.

In the experiments with toy balloons, the distance passed through was measured by means of a kinetograph and the time by means of a flywheel on ball bearings. An attached hundred-division scale and vernier were photographed by the kinetograph, the speed of the balance-wheel being determined by comparison with a pendulum clock by means of an electric chronograph.

It is difficult to determine the lifting force, since it varies during the start. The variability is due to the fact that the balloon, strongly heated by radiation before the start, is cooled and contracted by the air coming in contact with it.

For the temperature of the experimental balloon before the start there is radiation equilibrium, the same as for the temperature of a free balloon in equilibrium.* For the latter the

* Emden: "Zur Füllungstemperatur eines Freiballons." Zeitschrift für Flugtechnik und Motorluftschiffahrt, 1912, p.315.

temperature influences have been thoroughly investigated, on account of their practical importance.* In many free-balloon trips, Schmauss kept a continuous record of the temperature of the gas in the balloon and of the surrounding air. He found, in positions of equilibrium, the gas temperature, in agreement with the simultaneous indication of a black-bulb thermometer, up to 36°C. (96.8°F.) above the temperature of the atmosphere. He found the effect of the cooling, due to the motion of the balloon, to be less than had previously been assumed. Nevertheless, there remains the fact of the peculiar one-sided instability of a free balloon and its reversal by night, thus demonstrating the existence of a certain ventilation effect. The definite proof of this was furnished by the Göttingen toy balloon experiment No.11, in which the falling of the balloon from a position of equilibrium was photographed by a kinetograph. In this small-scale experiment, there could be no decrease in radiation due to the altitude. The starting experiment No. 7, showed still plainer the extraordinarily rapid ventilation effect. In this experiment the balloon started with a good lifting power, but its motion was reversed after it had risen hardly its diameter.

This difficulty in determining the lifting power was met in the following manner; In all experiments the balloon was kept out of the direct sunlight. In the pictures it always appeared dark, against a light background. Toy balloons have the very favorable

* Schmauss and Von Bassus, "Zur Gastemperatur des Freiballons" Zeitschrift für Flugtechnik und Motorluftschiffahrt, 1911, p.216.

characteristic that they offer but little resistance to inflation up to a certain volume, beyond which, however, any further increase in volume requires considerable increase in the inflation pressure. Furthermore, for the exact evaluation of certain experiments, they were made with as great a lifting power as possible. Lastly, the direct determination of the latter, by weighing before the start, was utilized exclusively for the very first part of the flight, about one-fourth of the balloon's diameter. In all other experiments, the lifting power was calculated from the vertical velocity attained by the balloon after a long time. The latter experiments, however, were only utilized for determining the flight path. Figs. 1a to 1c show the arrangement of the experiments and the range of the measurements.

The first arrangement, for experiments 1-9, covered a path of not quite 2 meters (6.56 feet) or about 20 balloon diameters. Before the start the balloon was held by an electromagnet and was released by breaking the electric circuit.

The experiments showed a deviation of the balloon from the vertical. At first this was supposed to be due to air currents. Accordingly all the doors and windows were made tight by stopping up the cracks. Then the air currents for different positions of the sun were tested by means of tobacco smoke and the best place in the room for the experiments was selected accordingly. It was found, after a half-hour of perfect quiet after completing the preparations for the experiment, that the only motion of the air was a to and fro oscillation, similar to that of water in a basin. The oscillation period varied from one to several minutes, the amplitude being the greatest at about half the height of the room where it was about 20 cm (7.87 in.). This gave a maximum vertical velocity of about 2 cm (0.79 in.) per second. In the portion of the room where the experiments were performed, the vertical velocity was only a small fraction of the above.

In the second series of experiments, embracing Nos. 10-21, the air was tested by smoke before each experiment and the start made at the instant when the air motion had reached a culmination point. The scale of the picture was considerably increased in comparison with the first arrangement, so that the utilizable portion of the flight path was about 8 balloon diameters. The balloon was photographed, not only from the front but also from the side, by means of a large mirror, so that on each film section both pictures show, as seen in Figure 1b.

The third arrangement reduced the measuring range to about half a balloon radius, 1 mm (.04 in.) on the picture corresponding exactly to 25 mm (.98 in.) at the experiment location. The mirror giving the side view of the balloon was removed. Every precaution was taken against air currents. Window and door cracks were stopped up. A half-hour rest was observed before each experiment, but the testing of the air before the experiment was omitted for the following reason.

It had already been found that the velocity of the air oscillations at the location of the experiments, even when the starting instant coincided with the maximum velocity, was so slight and the Reynolds number so small, that the yielding motion of the air about the balloon was purely laminar. The disturbance, therefore, did not spread out to any appreciable distance from the balloon, which found itself at the instant of starting in essentially quiet air, only moving upward very slowly and uniformly, conditions which did not impair the assumptions for a form of flow corresponding practically to the potential motion.

Among the hundred balloons on hand, one had been found in all previous experiments to have an especially good shape. Its rotational symmetry was perfect, no difference being discoverable in its horizontal diameters. It was, however, slightly elongated, its vertical diameter being about 2.5% longer than its horizontal diameter. This balloon was used for the principal experiments.

The lifting force was determined by means of a starting-balance. Above the knife of one beam, two insulated pieces of brass were attached so as to leave about a centimeter (0.4 in.) between them. An 0.8 mm (0.03 in.) hole was drilled through both brass pieces parallel to the knife-edge. Through these holes was passed a fine silver wire which was held in place by suitable wedges. This wire was passed through a loop of the thread holding the balloon, so that the melting of the wire would release the balloon. The electric circuit was completed from the balance post by means of two strips of sheet steel 0.05 mm (0.002 in.) thick and about 10 mm (0.4 in.) wide, whose stiffness scarcely affected the balance. For one of the balance pans there was substituted a circular aluminum disk suspended horizontally, which hung freely in a glass cylinder containing kerosene.* This effectually damped the oscillations of the balance almost to aperiodicity.

The balance was first tared without the balloon and then with the balloon attached. Enough weight was then removed from the scale-pan under the balloon for equilibrium to be restored by the loss of gas during a full half-hour. This period allowed the air in the experiment room to come to rest. The balance was made somewhat more stable than previously, so that the balloon end sank slowly without oscillations. Immediately before the start, the position of the pointer was read and, after experiment 41, with the aid of a telescope.

Evaluation of the Göttingen Experiments.- From the starting experiments of the second arrangement with toy balloons, the flight path was determined as follows: On each photograph the cartesian coordinates were measured with ruler and lens. The location of the balloon center was considered as being halfway between the upper and lower and the right and left edges of the balloon. The greatest errors were thus eliminated (See appendix of this paper). On the other hand, the individual measurements of the balloon edge were accurate only to about 0.1 mm (0.004 in.).

* The position of the disk represents the balance in its arrested position. In this position the vessel is closed by a glass cover, which is connected with the damping disk and is therefore raised with it, when the balance is freed by raising the middle knife.

The direct photographs gave the X and Z coordinates. The side view, obtained by means of the mirror, gave the Y coordinates. The values x, y and z, thus determined, were made non-dimensional by dividing by the radius of the balloon. The latter was found by measuring the individual pictures and separately for the mirror pictures. The somewhat different scale of the latter was also taken into consideration.

The non-dimensional coordinates, thus obtained, are plotted in Figs. 2a - 2c, first in the ground-plan and then (on the basis of the latter) in elevation.

The acceleration and relative apparent mass were determined from the third series of experiments in the following manner. The path followed by the center of the balloon was determined by projecting the picture on millimeter paper. The scale adopted after completing the experiment was determined in the same manner and the path plotted accordingly.

The volume for the determination of the displacement D was computed as that of an ellipsoid. For this purpose, before beginning the experiments, two horizontal projections were made at right angles to each other, from which three diameters perpendicular to one another could be determined. Since all the exactly evaluated experiments were made with the same balloon, its constancy of volume was assumed for all these experiments. Its volume was, therefore, considered to be proportional to the third power of the vertical diameter. Still the displacements thus found were graphically represented as a function of the lifting forces determined by weighing.

The time was determined, in thousandths of a revolution, by the position of the time wheel or dial on every picture. The velocity in the interval between two photographs was determined as the quotient of the differences and ascribed to the mean time between the two photographs. Thus there was obtained a series of velocities and corresponding time intervals. From this series, after plotting graphically, the acceleration was determined by the method of the smallest squares. This graphic always gave a good line (Fig. 3 is such a graphic), with the single exception of experiment No. 63, which should be eliminated anyway, on account of its small lifting power, due to the ventilation effect (See Section 2).

Hence, neither the location of the balloon before the start, nor the instant of the start was employed in order to avoid errors, which could have arisen from slow starts or from the slight vertical velocity already possessed by the balloon before the start, due to the motion of the balance.

The velocity of the time wheel was considered uniform during the experiment and determined as follows: The time wheel marked its complete revolutions on the chronograph paper, a pendulum clock marked whole seconds and the fusing current marked the instant of start. The first two series of marks gave the velocity of the wheel for the middle of each second. This velocity was plotted in right-angled coordinates as a function of the time and therefrom the velocity at the instant indicated by the starting mark was determined. The peripheral velocity of the time wheel, thus determined, was further provided with a correction factor, which took into account the unevenness of the scale divisions and which, after a calibrating once for all, was determined from the position of the wheel at the instant of starting.

Therefrom the acceleration, hitherto calculated for the thousandth of a revolution of the wheel, could be recalculated for a second as the unit of time.

The acceleration b thus gave the relative apparent mass according to the formula

$$k = \frac{P}{D}\left(\frac{g}{b} + 1\right) - 1.$$

The results of experiments 53 to 62 are combined in Fig. 3. Only such experiments were omitted as could not be evaluated.

4. Direct Experimental Results.- The first series of experiments with toy balloons must be considered as preliminary. These experiments demonstrated the tendency of the starting balloon to forsake its vertical course. This deviation was first thought to be the effect of disturbing influences, probably air currents, for which reason the measures already described were undertaken for the elimination of such influences in the following experiments.

The second series of experiments demonstrated, first, that the lateral deviation of the balloon was not the result of disturbing influences, but was an inherent characteristic of the motion of the sphere.* The path curves show deviations in entirely different directions in the experiment room, thus eliminating the conjecture that the mirror might exert any appreciable influence.** In the second place, these experiments showed the sudden beginning of the deviation after previous vertical ascent. Thirdly, the path curves show that the deviation begins after the balloon has ascended four or five times its diameter and is, therefore, independent of the lifting force. Fourthly, it is observed that the balloon, after leaving the vertical, follows an even course for some distance. Subsequently double curves are formed without seeming to follow any law. The conclusions reached by comparing the various curves are strengthened by the fact that the experiments were performed with different balloons.

The results obtained by the third experimental arrangement are plotted in Fig. 3. The demonstration of the availability of the kinetograph for acceleration measurements is to be regarded as one result of these experiments. The greatest disregarded source of errors must be sought in the flexibility of the contact springs on the time wheel.

No dependence of the relative apparent mass k on the rel-

* The path curves are shown in Figs. 2a - 2c. The camera was located in the direction indicated by the arrows on the ground plan.

** This action would consist in the attraction of the balloon toward the wall. See "Flüssigkeitsbewegung" by Prandtl: Handwörterbuch der Naturwissenschaften, Vol.IV, p.136, Jena, 1913.

ative lifting force P/D can be discovered. The values of k lie near the number 0.5, which applies to the potential motion. The agreement is still better after making the two following corrections.

We must calculate the effect of the deviation from the spherical shape of the toy balloon used for the principal experiments. We must also estimate roughly the effect of the viscosity of the boundary layer of air on the result.

5. Apparent Mass of a Rotation Ellipsoid. Approximation Formula for nearly Spherical Rotation Ellipsoids.- As already mentioned, the two horizontal diameters of the balloon were of the same length and the vertical diameter was about 2.5% longer. The experimentally determined value of k does not, therefore, hold strictly true for a sphere, but for a body which may be regarded as a slightly elongated rotation ellipsoid.

The principle is known for the calculation of the potential flow around an ellipsoid of the half-axis a, b, c (The calculation is given in Lamb's "Lehrbuch der Hydrodynamik," pp. 179-180, Leipzig, 1907). For the motion of an ellipsoid in the direction of its a-axis, the apparent mass is found to be

$$k = \frac{\alpha}{2 - \alpha} \qquad (1)$$

in which

$$\alpha = abc \int_0^\infty \frac{d\lambda}{(a^2+\lambda)\sqrt{(a^2+\lambda)(b^2+\lambda)(c^2+\lambda)}} .$$

For a rotation ellipsoid $b = c$, hence

$$\alpha = ac^2 \int_0^\infty \frac{d\lambda}{(a^2+\lambda)^{\frac{3}{2}}\,(c^2+\lambda)} \tag{2}$$

The undetermined integral is

$$\frac{2\,a\,c^2}{a^2-c^2}\left(\frac{1}{\sqrt{a^2+\lambda}} + \frac{1}{2\sqrt{a^2-c^2}}\log\frac{\sqrt{a^2+\lambda}-\sqrt{a^2-c^2}}{\sqrt{a^2+\lambda}+\sqrt{a^2-c^2}}\right) \tag{3a}$$

Between the limits 0 and ∞ it gives

$$\alpha = \frac{\frac{c^2}{a^2}}{1-\frac{c^2}{a^2}}\left[\frac{\log\left(1+\sqrt{1-\frac{c^2}{a^2}}\right)-\log\left(1-\sqrt{1-\frac{c^2}{a^2}}\right)}{\sqrt{1-\frac{c^2}{a^2}}} - 2\right] \tag{4a}$$

The undetermined integral may also be written:

$$-\frac{2\,a\,c^2}{c^2-a^2}\left(\frac{1}{\sqrt{a^2+\lambda}} + \frac{1}{\sqrt{c^2-a^2}}\,\mathrm{arc\,cos}\,\frac{c^2-a^2}{c^2+\lambda}\right) \tag{3b}$$

and therefore

$$\alpha = 2\,\frac{c^2}{c^2-a^2}\left(1-\frac{\mathrm{arc\,cos}\,\frac{a}{c}}{\sqrt{\frac{c^2}{a^2}-1}}\right) \tag{4b}$$

The expressions 3a and 3b are identical, as likewise the formulas 4a and 4b. When introduced into equation 1, they give:

$$k = \frac{1-\frac{c^2}{a^2}}{1-\frac{1}{2}\frac{c^2}{a^2}\,\frac{\log\left(1+\sqrt{1-\frac{c^2}{a^2}}\right)-\log\left(1-\sqrt{1-\frac{c^2}{a^2}}\right)}{\sqrt{1-\frac{c^2}{a^2}}}} - 1 \tag{5a}$$

or

$$k = \frac{\frac{c^2}{a^2} - 1}{\frac{c^2}{a^2} \frac{1}{\sqrt{\frac{c^2}{a^2} - 1}} \operatorname{arc} \cos \frac{a}{c} - 1} - 1 \tag{5b}$$

Either expression, 5a or 5b, can be converted into the other. The former is adapted to elongated and the latter to flattened rotation ellipsoids. For an extremely elongated ellipsoid, $k = 0$. For a sphere or circular disk, the known results are obtained.

In order to obtain a convenient approximation formula for such ellipsoids, which differ but little from spheres, the logarithm in the formula 5a is converted into a series according to the powers of $1 - \left(\frac{c}{a}\right)^2$ and then the quantity $1 - \frac{c}{a}$ is treated as "small", so that its higher powers may be disregarded. We thus obtain the approximation formula

$$k = \frac{3}{5} \frac{c}{a} - \frac{1}{10} \tag{6}$$

This formula may be likewise obtained from formula 5b.

In the balloon under consideration, $\frac{c}{a} = 0.975$, which gives k a value of 0.485. The deviation from the number 0.5 for the sphere is negligible.

<u>6. Estimation of the Friction</u>.- In order to estimate the effect of the friction of the air on the experimentally found value of k, the friction on the top of the sphere is replaced by the friction on the convex surface of a cylinder moving at

right angles to its axis, both the diameter and length of the cylinder being the same as the diameter of the sphere. The surface areas of both are exactly the same.

One objection to the comparative method of approaching the two-dimensional problem shall immediately be met. For a rotation cylinder, the potential flow gives a relative apparent mass $k = 1$, twice as great as for a sphere.* This great difference is very significant for the different effect of a spherical and an infinitely broad obstacle on a flowing fluid. It is due to the hydrodynamic action throughout the whole space, while the mechanics of friction operate exclusively in the immediate vicinity of the surface. Any such difference in the friction is therefore not to be feared. Blasius calculated the friction of a cylinder according to Prandtl's boundary-layer theory (Prandtl: "Grenzschichtentheorie." - See footnote (*) on Page 5). He found a frictional force of

$$4\, r\, l \sqrt{\pi \rho \;\; \eta\, t} \cdot b \qquad (7)$$

for a uniform acceleration b of a rotation cylinder of radius r and length l. This obstructing force is accordingly proportional to the square root of the time elapsed from the beginning of the motion. This would necessitate a reduction of the accel-

* Blasius: "Grenzschichten in Flüssigkeiten mit sehr kleiner Reibung," Zeitschrift für Mathematik und Physik, 1908, No.1. Blasius obtains $k = 2$ for the rotation cylinder, by disregarding the acceleration of his system of coordinates in the calculation of the acceleration for the formula

$$-\frac{dp}{ds} = \rho\,\frac{dw}{dt}\,.$$

After correcting the calculation of the acceleration, his method also gives $k = 1$.

eration and hence a concave curve toward the time axis in plotting the velocity against the time.* This cannot be established in spite of the smooth course of this curve.

Friction has the same effect on the mathematically determined value of the acceleration, as a constant force of the magnitude of the mean value R_m of the frictional force during the experiment, which may be calculated from the start for this rough estimate. This mean value of the time is then

$$R_m = 4\ r\ l\sqrt{\pi\ \rho\ \eta}\ \frac{1}{T}\int_0^T \sqrt{t}\ d\ t \cdot b = \frac{8}{3}\ r\ l\sqrt{\pi\ \rho\ \eta\ T} \cdot b$$

Since $l = 2\ r$ for the cylinder under consideration,

$$R_m = \frac{16}{3}\ r^2\sqrt{\pi\ \rho\ \eta\ T} \cdot b \tag{8}$$

This obstructing or damping force has the same effect as a like reduction in the lifting force. It increases the experimental value k of the relative apparent mass by an amount d in comparison with its magnitude for a frictionless start, for which the calculation gives

$$d = \frac{g}{b}\,\frac{R_m}{D}$$

If we substitute, in the above formula, the value of R_m according to equation (8) and express the displacement D as the radius of the sphere, we obtain

$$d = \frac{4}{\sqrt{\pi}}\ \frac{1}{r}\sqrt{\nu\ T} \tag{9}$$

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* This application was made for all the evaluated experiments. Figure 4 is an example.

Formula 9 enables the calculation of d for every experiment. The balloon radius was introduced according to the results of the measurements in each experiment. The start was determined by extrapolation from the diagram of the velocities (Fig. 4) and the time of the last picture used for exact evaluation was taken as the end of the experiment.

For the kinematic viscosity $\nu = \frac{\eta}{\rho}$, the value 0.157 cm²/sec. was introduced, which was determined from the temperature of the air (23.1°C) and the barometer reading (751.5 mm) on the day of the experiment (Prandtl: "Flüssigkeitsbewegung," p.116. - See footnote on Page 15 (**). The calculation gives the following values (d) of the effect of friction in the individual experiments.

Exp. No.	53	54	57	58	60	62
d =	0.048	0.049	0.052	0.060	0.069	0.075

The effect of friction is therefore from three to five times as large and of opposite sign from that of the somewhat elongated balloon and, like the latter, too small in comparison with the experimental errors to be substantiated.

Nevertheless, the points representing the actual experimental results, lie close to the curve passing through the points which were to be expected as experimental results from the above calculated value k = 0.485, taking into account the friction, on the basis of the foregoing calculation. This curve is shown on Fig. 3. It has no mathematical nor physical significance.

7. The Start.- The following discussion of the process of starting presupposes not only rotation symmetrical arrangement of the masses, but also coincidence of the center of gravity with the center of volume of the sphere.

The start depends, therefore, on the following constants, which are independent of one another: the density (ρ) of the fluid, the viscosity (η) of the fluid, the mass (M) of the sphere, the motive force (P) and the radius (r) of the sphere. From these constants two non-dimensional quantities, $\frac{r^3\rho}{M}$ and $\frac{P\rho}{\eta^2}$, can be obtained.

The two following conversion factors,

$$g = \frac{P}{\rho V - M} \quad \text{and} \quad D = P \frac{\rho V}{\rho V - M},$$

in which V represents the volume of the sphere, will now be defined.

When the motive force is hydrostatic, these quantities represent the weight of the unit mass (g) and the weight of the fluid displaced (D). ρ, η, D, P and g may therefore be regarded as independent constants.

Since the case of hydrostatic ascending powers is one of the most important and especially since all the experiments were carried out with such motive forces, this method of presentation will be adhered to in what follows. Since, in the transition from the first-mentioned system of independent constants to the last-mentioned system of constants, their number remains unchanged, it is demonstrated, by the establishing of the above conversion formulas, that the universal application of all the considerations is not impaired by the assumption of hydrostatic motive forces. From the last five constants both the following non-dimensional quantities can be formed

$$S = \frac{P}{D} \quad \text{and} \quad Q = \frac{D\rho}{\eta^2} ,$$

The velocity v, attained by the sphere after a long time, is given by the non-dimensional $\psi = \frac{1}{\pi} \frac{P}{r^2 v^2 \rho}$, which is a function of Reynolds number $R = \frac{2 r v \rho}{\eta}$. Between the two latter, the same as between the aforementioned/non-dimensional quantities, there is the relation $R^2 \psi = \frac{\pi}{4} Q S$. The quantity S may therefore be called the "relative lifting force."

For the case of hydrostatic lifting forces, it is still to be remarked that the quantity g does not explicitly occur in the two non-dimensionals S and Q, but is contained as a factor in $D = g \rho V$ and also in P. It is therefore contained in the non-dimensional Q, but not in S. While the start, as a whole, is a very complex process, various simplifying assumptions give such motions as describe the individual stages of the starting process in an essentially correct manner. The more successful we are in augmenting the number of these stages (either by theories or experimental results) just so much the deeper will be our knowledge of the starting process itself, even though it can not be covered by a comprehensive theory. Thus far, four such stages have been distinguished, the first two of which are sufficiently well known.

The first agrees with the potential motion at some distance from the sphere. On the surface of the sphere there is developed a boundary layer of increasing thickness, from which there accumulates behind the sphere an increasing amount of "dead" air.

The second stage begins with the freeing of the airflow from the surface of the sphere. The dead air changes to a ring of vortices slowly receding from the sphere. The flow diagram is still rotationally symmetrical, though this symmetry rapidly becomes increasingly unstable. Toward the end of the second stage, the rectilinear motion seems to be stable toward slight disturbances, but unstable toward greater disturbances, whereby magnitude of the disturbance would decrease with the time, according to a definite law.

The third stage begins with the destruction of this unstable rotational symmetry. Its beginning is not sharply defined and takes place somewhat earlier or later in different cases. The definitive deviation first begins when any chance disturbance reaches the critical value, which decreases with the lapse of time. Furthermore the deviation begins in accordance with an exponential law, which takes effect with the first small deviation and is therefore, during this time, proportional to the accidental magnitude of the disturbance. The transition of the exponential law into the real deviation takes place therefore just as much sooner as the small initial disturbance begins. The path of the balloon center is practically straight in the second stage and the flow is therefore still symmetrical. This experimental result is due to the fact that, immediately after the beginning of the deviation, a motion reigns similar to the one subsequently described, which is, however, not yet perfectly periodical like the latter, but nevertheless possesses great stability. The latter prevents slight subsequent disturbances from producing so great effects as the disturbances which caused the deviation.

The fourth stage is the "motion after a long time." This is not chiefly dependent on the initial conditions and therefore applies not only to the starting process, but to every motion of a sphere under the influence of a constant force. The "motion after a long time" may assume very diverse forms. It depends, like the starting process in general, on two non-dimensional quantities, at least one of which must contain the viscosity constant of the fluid. Aside from the motion-form of Stokes, there is one in which sometimes larger and sometimes smaller portions of the dead air behind the sphere are carried away in non-periodical succession and form a vortex trail of irregular structure. The center of the sphere then has a practically uniform, rectilinear motion.

One form of "motion after a long time," very different from the above, gives, as the path of the center of the sphere, a regular uniform curve around a vertical axis. This form can be especially well observed by means of air bubbles of about three millimeters diameter rising in water. The wave form is strikingly regular and the plane, in which the motion takes place, is strongly adhered to.* Manifestly, the motion is very stable toward disturbances. Its regularity gives rise to the supposition that, in contrast with the rectilinear motion, the undulatingly moving sphere leaves behind it in the fluid a regular and periodically distributed motion. In this, as in many other relations, the wave-like motion suggests the vortex series investigated by Karman.

The center of the sphere moves, after a long time, either in a vertical line or undulatingly around it. This straight line does not ordinarily pass through the point of origin of the motion. In the Göttingen starting experiments with toy balloons, it was found throughout to be removed in the direction of the initial lateral deviation from the starting point.

Summary.- At the start, according to the Göttingen experiments, the relative apparent mass of the sphere is 0.5, thus agreeing with the calculated result for the potential motion. After the release of a vortex ring, there follows a practically regular lateral deviation, whose intensity characterizes it as the upsetting of an unstable condition. The "motion after a long time" may follow in two different forms. Air bubbles in water follow regular undulatory paths with great stability.

* Small bubbles of one to two millimeters diameter seldom rise spirally, still smaller ones always vertically, while in bubbles with diameters of more than about 4.5 mm (.177 in.) the hydrodynamic pressure overcomes the surface tension and gives the bubbles laterally varying shapes, instead of spherical.

The theoretical "apparent mass" was calculated for a rotational ellipsoid. Special tests demonstrated that measurements of sufficient accuracy for calculating the accelerations can be made by means of a kinetograph. Information on work with kinetographs is given in the Appendix.

APPENDIX.

Work with the kinetograph.- The experiments described in this paper give abundant data on the suitability of kinetograms for the quantitative analysis of motions. For this purpose, the production of positive films is not generally necessary.

The employment of the kinetograph is indicated whenever a series of simultaneous positions of two or more points is to be determined. A special case of this nature is presented when a series of successive positions of a body is to be established, since each of these positions is determined from three (in simple cases from two) points. The determination of a series of positions of one or more points and the corresponding times forms another special case, which is probably the most frequent and important. An experimental arrangement without a kinetograph is often to be preferred, since it is troublesome of manipulation and gives results of limited accuracy. It use is inexpedient, for instance, when only the path of a point is desired.

As an example of the motion analysis of several points, take the experimental arrangement described in Section 2, for the simultaneous photographing of the same motion from two directions at right angles to each other. This was accomplished by means of a large mirror, so that both pictures appeared on the same negative (Fig. 1b).

In so far as possible, ordinary kinetographs should be employed. In any event, it is desirable to employ ordinary films, even when a special camera is built. The films are 35 mm (1.38 in.) wide and may be procured either with or without perforations. Normally there are 208 perforations per meter, but many makes differ slightly. Commercial cameras take 52 pictures about 19 × 24 mm (0.75 × 0.94 in.) on one meter of film. The normal number of exposures per second is 15, which may be increased, without special provision, to something over 20, but any considerable increase, like doubling the number, can not be demanded of the camera.

An especially firm mounting of the camera best facilitates the work. It must never be assumed, however, that all the pictures are oriented exactly the same with respect to the perforations, the edge of the film or the edge of the picture. If the finished film is run through a kinetoscope, the individual pictures will always show lateral displacement. For making measurements, it is therefore essential to include fixed marks (usually two) in the photograph. If only the rectilinear motion of a point is to be tested, one mark will suffice, which is then placed in the direction of the motion of the point.

The scale of the pictures may be determined by photographing, either before or after the experiment, a ruler placed in exactly the right position. Of course any different adjustment of the focus or shutter, for this purpose, is inadmissible. One of the negatives on which the ruler is included can then be cut out and used directly as a scale for measuring the individual photographs. The measurements may be still better facilitated by running the film through the kinetograph again after the experiment and thus showing the ruler on each picture, or, still better, by optically producing a real image of the ruler during the experiment and including it in the photographs. In the latter case only, the photographed ruler will be oriented the same on all the pictures, with respect to the photographed space, and thus simultaneously serve as a fixed mark in the sense of the above-mentioned requirement, so that only one measurement on each picture determines the position of the moving object.

In this way not only the scale of the picture is determined, but possible distortions due to defects of the lens are eliminated, which does not then need to be absolutely faultless, but simply capable of giving a clear picture within the experimental field. The last one of the above-mentioned methods of photographing the ruler, however, introduces the errors of a second optical system.

In making the measurements, a strong lens usually suffices to bring out anything in the photograph that may lack clearness. By means of a 15-power magnifying lens and a finely divided scale, accurate measurements can be made to within 0.01 mm (.0004 in.). Microscopes for film measurements must have a special damping device. Measuring by projection on millimeter paper has not proved successful.

The best possible illumination is desirable for taking the picture. It does no harm, even if the different marks were formed by sunlight reflected directly into the camera. Good illumination means accuracy, since, in order to obtain accurate pictures, not only must the ratio of the length of exposure to the time interval between the exposures be as small as possible, but also the shutter aperture. The latter is much more necessary for kinetograms than for ordinary photograms. The film never lies perfectly flat, but curves more or less, either forward or backward. This cannot be remedied by sharp focusing, but only by reducing the aperture.

The marks, especially the movable ones, should not be formed by division lines between light and dark, but by symmetrical dark lines on a light background or vice versa. Such lines may be very faint in the picture and still be easily deciphered. The use of a vernier has proved very successful for faint lines. The coincidence between the stationary and therefore sharp lines of the ruler and the moving faint marks of the vernier (or vice versa) can be easily and accurately determined.

The time can be measured by adjusting the speed of the camera so accurately as to give a uniform known interval of time between successive exposures. This, however, would usually be difficult and only attainable by mechanical means and would also necessitate the running of the kinetograph a long time before the beginning of each experiment. The alternative consists in simultaneously photographing some object whose motion is known. A simple and reliable method, only somewhat inconvenient to evaluate, consists in simultaneously photographing a falling lead weight. The object with known motion can be a pointer revolving in front of a dial. It is better to let the dial rotate and the pointer remain stationary. In this way one avoids the parallax due to the space between the dial and the pointer and also saves space on the photograph by showing only the portion of the dial behind the pointer. What has been said regarding marks also applies to the dial and pointer. A vernier is of advantage.

Fig. 1c shows the time wheel or dial, the scale and the vernier attached to the stand. If the dial had been placed near the ascending balloon, it would have disturbed the air. Hence it was placed near the kinetograph and facing the balloon, while a mirror placed midway between the balloon and the kinetograph gave an image of the dial near the balloon.

The speed of the time wheel or dial may be regulated by clockwork, brake or other device and thus be a real timepiece, or it may be only a flywheel, all of whose revolutions are compared with the readings of a contact clock by means of a chronograph. The following type of such a dial has given good results. A heavy lead pipe was fitted to the rim of the rear wheel of a bicycle and then smoothed off. The sprocket was replaced by a sliding contact which gave one contact for every revolution. The ball bearing shaft was screwed to a strong support, a hundred-division scale was fastened to the wheel and accurately balanced. (In the experiments under consideration, the starting instant was thus determined, which must be known, in order to interpret the chronogram.) It is important to make the contact arrangement on the time dial so that the contact will be made every time in exactly the same position, unaffected by elastic deformation, rust, or dirt. The signals given by the sliding contact are compared with the signals given by a contact clock (preferably a pendulum clock) by means of an electric chronograph. If such a clock is available, it is advisable to obtain one with three or four writing levers, thus rendering it possible to determine any other desirable time points, e.g., the beginning and end of the experiment.

Another arrangement of the time dial is close in front of the film. In this case it is made very small and its divisions are indicated by holes or slots. This time dial or wheel is driven by means of a shaft which enters the camera from the rear. The shaft is driven either by clockwork or a flywheel with contact. This time wheel leaves marks on the film by means of its shadows. The illumination may come through the lens by making the corresponding portion of the background of the picture light, or it may be direct, in which case the portion of the shutter opening behind the time wheel must be separated from the rest by an opaque partition.

In acceleration measurements, the acceleration of the time wheel must sometimes be taken into account, in addition to its speed.

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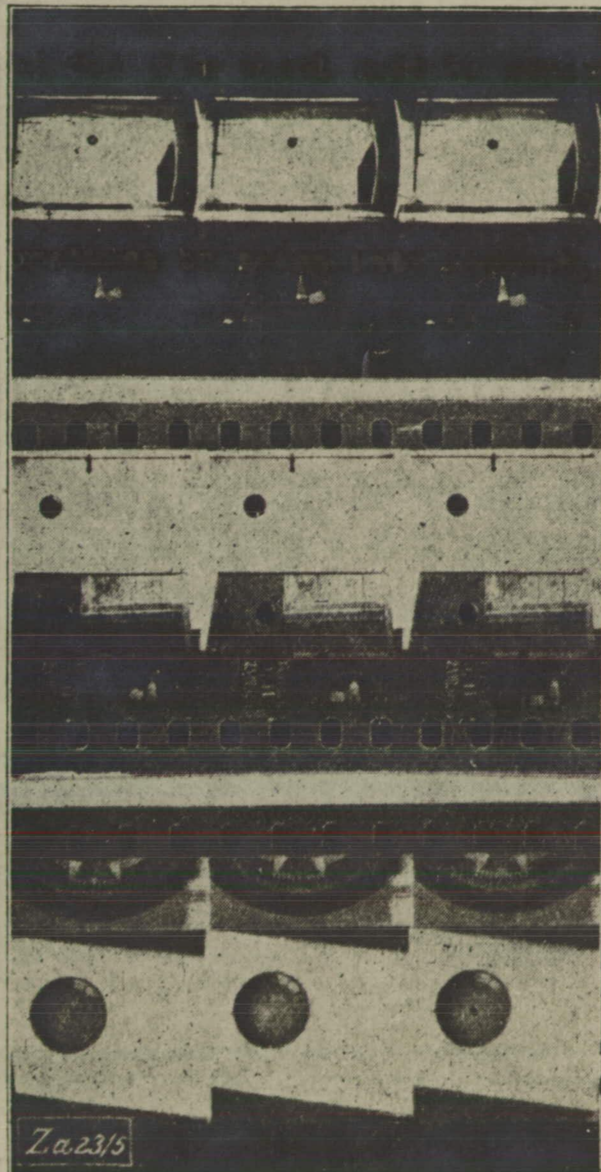

Fig. 1a

Fig. 1b

Fig. 1c

Film pictures for the three series of experiments

Figs.2a - 2c.

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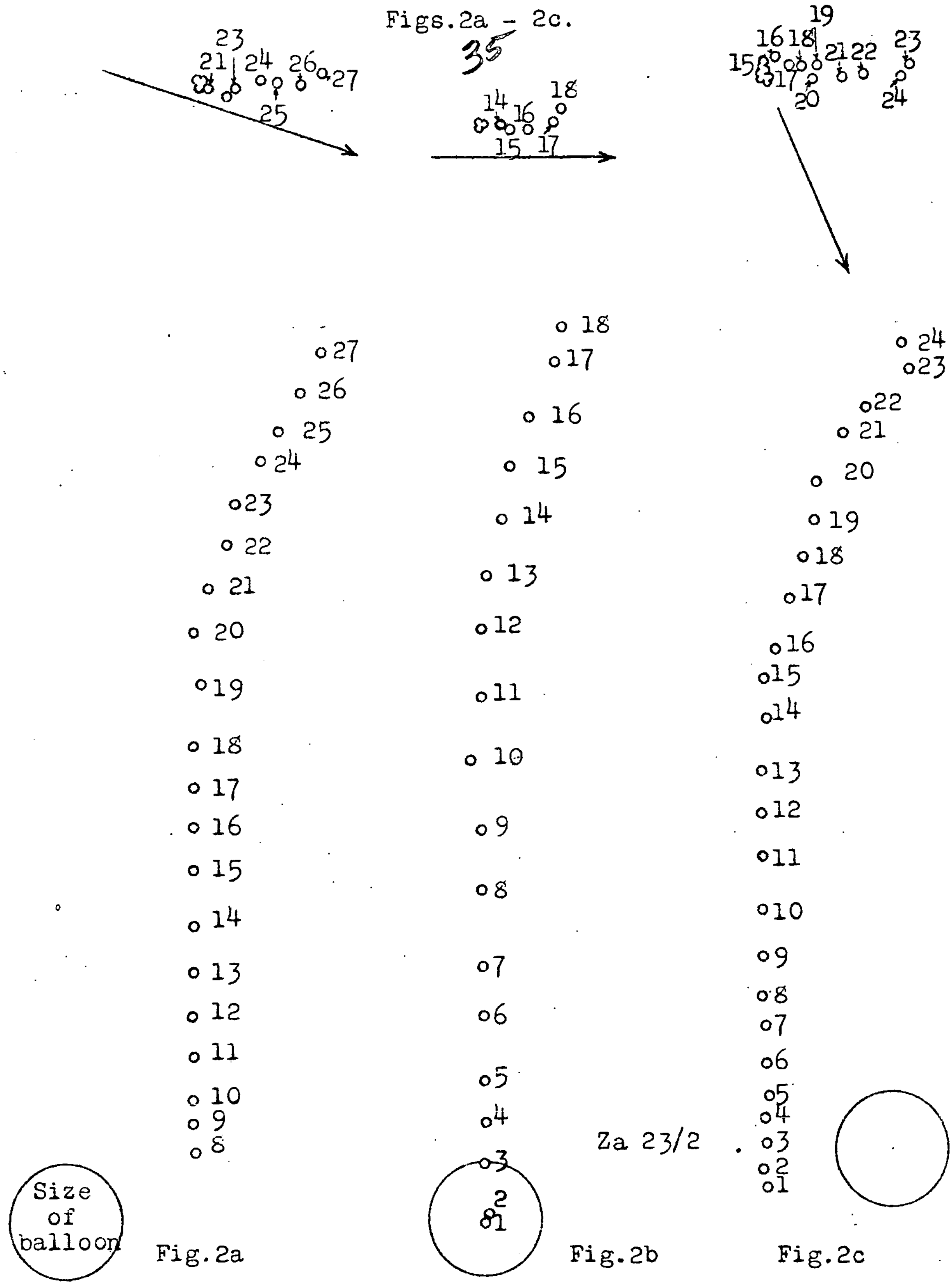

Fig.2a Fig.2b Fig.2c

Experiment 15. Experiment 18. Experiment 19.

Path of balloon center in plan and elevation. Arrows point to kinetographs.

Figs.3.& 4.

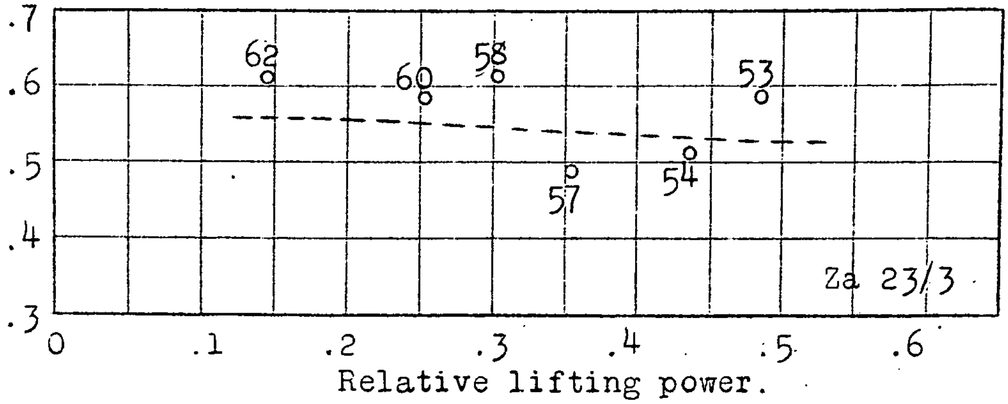

Fig.3. Relative apparent mass from experiments 53-62

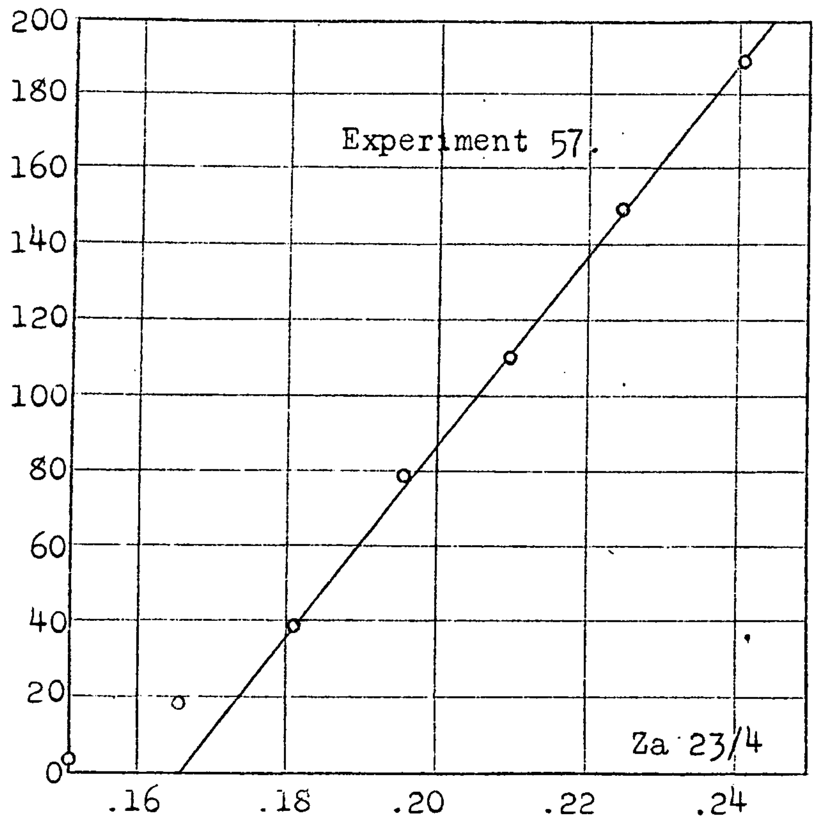

Fig.4. Velocity plotted against time.